Project systems theory – Solutions

Final exam 2018–2019, Thursday 24 January 2019, 9:00 – 12:00

Problem 1

(2 + 4 + 9 = 15 points)

Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_1x_2 \\ -x_1 + x_1^2 + x_2^2 + u \end{bmatrix}, \qquad y = 2x_1x_2 \tag{1}$$

with state $x = [\, x_1 \;\; x_2 \,]^{\mathrm{T}}$, input $u \in \mathbb{R}$, and output $y \in \mathbb{R}$.

(a) Below, we will frequently use the shorthand notation

$$f(x,u) = \begin{bmatrix} x_2 - 2x_1x_2 \\ -x_1 + x_1^2 + x_2^2 + u \end{bmatrix}, \qquad h(x) = 2x_1x_2, \tag{2}$$

such that the dynamics (1) can be written as

$$\dot{x} = f(x,u), \qquad y = h(x), \tag{3}$$

To show that

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

is an equilibrium point for $u(t) = 0$, $t \geq 0$, compute $f(\bar{x}, 0)$ to obtain

$$f(\bar{x}, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5}$$

Hence, $\bar{x}$ is an equilibrium point (for $u(t) = \bar{u} = 0$, $t \geq 0$).

(b) Equilibrium points $\bar{x}$ corresponding to the constant input $u(t) = \bar{u} = 0$ are obtained as solutions of

$$0 = f(\bar{x}, \bar{u}). \tag{6}$$

Substituting $\bar{u} = 0$ and using the definition (3), we obtain

$$0 = \begin{bmatrix} \bar{x}_2 - 2\bar{x}_1\bar{x}_2 \\ -\bar{x}_1 + \bar{x}_1^2 + \bar{x}_2^2 + 0 \end{bmatrix}, \tag{7}$$

where $\bar{x} = [\, \bar{x}_1 \;\; \bar{x}_2 \,]^{\mathrm{T}}$. The first element yields

$$\bar{x}_2 - 2\bar{x}_1\bar{x}_2 = \bar{x}_2(1 - 2\bar{x}_1) = 0, \tag{8}$$

such that $\bar{x}_2 = 0$ or $\bar{x}_1 = \frac{1}{2}$. Substituting $\bar{x}_2 = 0$ in the second element of (7), we get

$$-\bar{x}_1 + \bar{x}_1^2 + 0 + 0 = \bar{x}_1(\bar{x}_1 - 1) = 0, \tag{9}$$

such that this leads to the equilibrium points

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{10}$$

Note that the second equilibrium point was already given in (4). Similarly, substituting $\bar{x}_1 = \frac{1}{2}$ in the second element of (7) leads to

$$-\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \bar{x}_2^2 + 0 = -\tfrac{1}{4} + \bar{x}_2^2 = 0 \tag{11}$$

such that the final two equilibrium points are given as

$$\bar{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \bar{x} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}. \tag{12}$$

(c) In order to find the linearized dynamics around the equilibrium point given by $\bar{x}$ and $\bar{u}$, define the perturbations

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{13}$$

Then, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t), \tag{14}$$

after which it can be concluded from (2) that

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} -2x_2 & 1 - 2x_1 \\ -1 + 2x_1 & 2x_2 \end{bmatrix}. \tag{15}$$

Evaluation of the result at $(\bar{x}, \bar{u})$ gives, after substitution of (4),

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{16}$$

Similarly, it is easy to see that

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{17}$$

Finally, denoting the nominal output $\bar{y}$ as

$$\bar{y} = h(\bar{x}) = 0, \tag{18}$$

and defining the perturbation

$$\tilde{y} = y - \bar{y}, \tag{19}$$

we can write the linearized output equation as

$$\tilde{y}(t) = \frac{\partial h}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t). \tag{20}$$

Here, by the definition (2), we have

$$\frac{\partial h}{\partial x}(x, u) = \begin{bmatrix} 2x_2 & 2x_1 \end{bmatrix} \tag{21}$$

and

$$\frac{\partial h}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 2 \end{bmatrix}. \tag{22}$$

Problem 2 (20 points)

Consider the family of polynomials

$$\mathcal{P}(\lambda) = \left\{ \lambda^3 + \theta_2 \lambda^2 + a\lambda + \theta_0 \,\middle|\, 2a \leq \theta_2 \leq 3a,\, a \leq \theta_0 \leq 4a \right\} \tag{23}$$

with a a real number.

By Kharitonov's theorem, the set of polynomials (23) is stable if and only if the four polynomials

$$p_1(\lambda) = \lambda^3 + 3a\lambda^2 + a\lambda + a, \tag{24}$$

$$p_2(\lambda) = \lambda^3 + 2a\lambda^2 + a\lambda + 4a, \tag{25}$$

$$p_3(\lambda) = \lambda^3 + 2a\lambda^2 + a\lambda + 4a, \tag{26}$$

$$p_4(\lambda) = \lambda^3 + 3a\lambda^2 + a\lambda + a, \tag{27}$$

are all stable. It is clear that $p_1(\lambda) = p_4(\lambda)$ and $p_2(\lambda) = p_3(\lambda)$, such that stability of only two distinct polynomials has to be determined.

First, stability of $p_1(\lambda) = p_4(\lambda)$ is studied using the Routh-Hurwitz test. To this end, consider the following table:

	λ^3	λ^2	λ^1	λ^0	
$3a\times$	1	$3a$	a	a	
$1\times$	$3a$		a		
		$9a^2$	$a(3a-1)$	$3a^2$	(result of step 1)
$(3a-1)\times$		$9a$	$3a-1$	$3a$	(after division by a)
$9a\times$		$3-a$			
			$(3a-1)^2$	$3a(3a-1)$	(result of step 2)
			$3a-1$	$3a$	(after division by $3a-1$)

Recall that a necessary condition for stability of a polynomial is that all coefficients have the same sign. For the original polynomial p_1, this gives that $a > 0$. Applying the same reasoning to the polynomial obtained after step 1, we get $3a - 1 > 0$, i.e., $a > \frac{1}{3}$. We now have $a > \frac{1}{3}$, which enables the division by a after step 1 as well as division by $3a - 1$ after step 2. Now, under this condition, it is easily checked that the final polynomial $(3a-1)\lambda + 3a$ is stable. Thus, we have that $p_1(\lambda) = p_4(\lambda)$ is stable if and only if

$$a > \tfrac{1}{3}. \tag{28}$$

Following the same approach, the Routh-Hurwitz table of $p_2(\lambda) = p_3(\lambda)$ is given as:

	λ^3	λ^2	λ^1	λ^0	
$2a\times$	1	$2a$	a	$4a$	
$1\times$	$2a$		$4a$		
		$4a^2$	$a(2a-4)$	$8a^2$	(result of step 1)
$(a-2)\times$		$2a$	$a-2$	$4a$	(after division by $2a$)
$2a\times$		$a-2$			
			$(a-2)^2$	$4a(a-2)$	(result of step 2)
			$a-2$	$4a$	(after division by $3-a$)

As before, the original polynomial $p_2(\lambda) = p_3(\lambda)$ gives that $a > 0$ is necessary for stability, whereas the result of step 2 gives $a > 2$. Under the latter condition, it can be shown that the final polynomial $(a-2)\lambda + 4a$ is stable. Hence, $p_2(\lambda) = p_3(\lambda)$ is stable if and only if

$$a > 2. \tag{29}$$

Combining the results (28) and (29) gives, through Kharitonov's theorem, that the family of polynomials $P(\lambda)$ is stable if and only if

$$a > 2. \tag{30}$$

Problem 3 (6 + 4 = 10 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -2 & 8 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} x(t). \tag{31}$$

(a) The unobservable subspace is given as

$$\ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 4 & -8 & 4 \end{bmatrix}. \tag{32}$$

As all rows are scaled version of the first row, it is clear that the unobservable subspace has dimension $3 - 1 = 2$. A basis is given by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \tag{33}$$

such that

$$\mathcal{N} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}. \tag{34}$$

(b) By definition of the unobservable subspace, we have

$$y(t) = 0, \quad t \geq 0. \tag{35}$$

Problem 4

(4 + 4 + 4 + 12 + 6 = 30 points)

Consider the linear system

$$\dot{x}(t) + Ax(t) + Bu(t), \tag{36}$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & 1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \tag{37}$$

(a) Due to the block-diagonal structure of A, its spectrum $\sigma(A)$ is given by

$$\sigma(A) = \sigma(-3) \cup \sigma\left(\begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}\right). \tag{38}$$

Computation of the characteristic polynomial of the matrix in the lower right block gives

$$\det\left(\lambda I - \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 3 & 5 \\ -1 & \lambda + 3 \end{vmatrix} = (\lambda - 3)(\lambda + 3) + 5 = \lambda^2 - 4, \tag{39}$$

after which solving $\lambda^2 - 4 = 0$ yields

$$\sigma\left(\begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}\right) = \{-2, 2\}, \tag{40}$$

such that the total spectrum is

$$\sigma(A) = \{-3, -2, 2\}. \tag{41}$$

As there exist eigenvalues with positive real part, the system (36) is not (asymptotically) stable.

(b) To verify controllability, compute

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -7 & 4 \\ 2 & -5 & 8 \end{bmatrix}, \tag{42}$$

whose rank is immediately observed to equal 2. Hence, as the state-space dimension of the system (36) equals 3, the system is not controllable. Note that this could have been concluded immediately from the block-diagonal structure of A in (37) and the observation that the corresponding entry in B equals zero.

(c) For the system to be stabilizable, all unstable eigenvalues of A need to be controllable. From (a), we recall that $\sigma(A) = \{-3, -2, 2\}$, such that 2 is the only unstable eigenvalue.

Denote $\lambda = 2$ and consider

$$\begin{bmatrix} \lambda I - A & B \end{bmatrix} = \left[\begin{array}{ccc|c} \lambda + 3 & 0 & 0 & 0 \\ 0 & \lambda - 3 & 5 & 1 \\ 0 & -1 & \lambda + 3 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & -1 & 5 & 1 \\ 0 & -1 & 5 & 2 \end{array}\right]. \tag{43}$$

It is clear that

$$\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = 3, \tag{44}$$

which equals the state-space dimension. Thus, the system (36) is stabilizable.

(d) Note that the block-diagonal structure of A and the structure of B in (37) enables us to write

$$A = \begin{bmatrix} -3 & 0 \\ 0 & \bar{A} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix}, \tag{45}$$

with

$$\bar{A} = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{46}$$

Then, a transformation of the form

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \bar{T} \end{bmatrix} \tag{47}$$

with $\bar{T}$ nonsingular yields

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & \bar{T}^{-1} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \bar{T} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & \bar{T}^{-1}\bar{A}\bar{T} \end{bmatrix}, \tag{48}$$

$$T^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & \bar{T}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{T}^{-1}\bar{B} \end{bmatrix}. \tag{49}$$

Now, the problem becomes to find a nonsingular matrix $\bar{T}$ such that

$$\bar{T}^{-1}\bar{A}\bar{T} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \qquad \bar{T}^{-1}\bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{50}$$

To do so, compute the characteristic polynomial of $\bar{A}$ in (46) as

$$\Delta_{\bar{A}}(\lambda) = \det(\lambda I - \bar{A}) = \lambda^2 - 4, \tag{51}$$

where the result (39) is recalled. After denoting

$$a_1 = 0, \qquad a_2 = -4, \tag{52}$$

the characteristic polynomial can be written as

$$\Delta_{\bar{A}}(\lambda) = \lambda^2 + a_1\lambda + a_2. \tag{53}$$

Now, to construct the transformation $\bar{T}$, consider

$$q_2 = \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{54}$$

$$q_1 = \bar{A}\bar{B} + a_1\bar{B} = \begin{bmatrix} -7 \\ -5 \end{bmatrix} + 0 \tag{55}$$

to form

$$\bar{T} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -5 & 2 \end{bmatrix}. \tag{56}$$

By construction, this matrix satisfies (50) with a_1 and a_2 as in (52). This can easily be verified after computing its inverse as

$$\bar{T}^{-1} = \tfrac{1}{9} \begin{bmatrix} -2 & 1 \\ -5 & 7 \end{bmatrix}. \tag{57}$$

(e) Following a similar partitioning as before, we can write

$$F = \begin{bmatrix} 0 & \bar{F} \end{bmatrix}, \tag{58}$$

with $\bar{F} = [\, f_2 \;\; f_1 \,]$.

The matrices $A + BF$ and

$$T^{-1}(A + BF)T = T^{-1}AT + T^{-1}BFT \tag{59}$$

have the same eigenvalues by similarity transformation. Specifically, using the block-diagonal forms (46) and (58), we obtain

$$T^{-1}(A + BF)T = \begin{bmatrix} -3 & 0 \\ 0 & \bar{T}^{-1}\bar{A}\bar{T} + \bar{T}^{-1}\bar{B}\bar{F}\bar{T} \end{bmatrix}. \tag{60}$$

Note that this feedback leaves the eigenvalue -3 untouched. This is no problem, as -3 is within the desired spectrum of the closed-loop system. Thus, what remains is to find $\bar{F}$ such that

$$\sigma\left(\bar{T}^{-1}\bar{A}\bar{T} + \bar{T}^{-1}\bar{B}\bar{F}\bar{T}\right) = \{-1, -1\}. \tag{61}$$

To this end, denote

$$\tilde{F} = [\, \tilde{f}_2 \;\; \tilde{f}_1 \,] = \bar{F}\bar{T}, \tag{62}$$

and compute

$$\bar{T}^{-1}\bar{A}\bar{T} + \bar{T}^{-1}\bar{B}\bar{F}\bar{T} = \begin{bmatrix} 0 & 1 \\ \tilde{f}_2 - a_2 & \tilde{f}_1 - a_1 \end{bmatrix}. \tag{63}$$

Due to its companion form, the characteristic equation of the matrix can immediately be given as

$$\Delta_{\bar{T}^{-1}(\bar{A}+\bar{B}\bar{F})\bar{T}}(\lambda) = \lambda^2 + (a_1 - \tilde{f}_1)\lambda + (a_2 - \tilde{f}_2). \tag{64}$$

As we would like the closed-loop system matrix to have eigenvalues at -1, -1, the desired characteristic polynomial is

$$p(\lambda) = (\lambda + 1)^2 = \lambda^2 + 2\lambda + 1. \tag{65}$$

After equating (64) and (65) and the substitution of the values for a_1 and a_2 in (52), we find

$$\tilde{f}_1 = a_1 - 2 = 0 - 2 = -2, \tag{66}$$

$$\tilde{f}_2 = a_2 - 1 = -4 - 1 = -5. \tag{67}$$

To find the feedback matrix $\bar{F}$ (in the original coordinates), compute

$$\bar{F} = \tilde{F}\bar{T}^{-1} = \tfrac{1}{9}\begin{bmatrix} -5 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -5 & 7 \end{bmatrix} = \tfrac{1}{9}\begin{bmatrix} 20 & -19 \end{bmatrix}, \tag{68}$$

such that the full feedback matrix $F = [\, 0 \;\; \bar{F} \,]$ is given as

$$F = \tfrac{1}{9}\begin{bmatrix} 0 & 20 & -19 \end{bmatrix}. \tag{69}$$

Problem 5 (15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{70}$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$ for $m = 1$.

Let $w_1 \neq 0$ and $w_2 \neq 0$ be two linearly independent eigenvalues of A^{T} for the eigenvalue $\lambda \in \sigma(A)$ (recall that $\sigma(A) = \sigma(A^{\mathrm{T}})$). Equivalently, they are left eigenvectors for A, i.e.,

$$w_1^{\mathrm{T}} A = \lambda w_1^{\mathrm{T}}, \qquad w_2^{\mathrm{T}} A = \lambda w_2^{\mathrm{T}}. \tag{71}$$

Now, let

$$w = \alpha_1 w_1 + \alpha_2 w_2, \tag{72}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$ be a linear combination of the left eigenvectors. Then,

$$w^{\mathrm{T}} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = \begin{bmatrix} w^{\mathrm{T}}(A - \lambda I) & w^{\mathrm{T}} B \end{bmatrix}. \tag{73}$$

Note that

$$w^{\mathrm{T}}(A - \lambda I) = 0 \tag{74}$$

due to (71). Moreover,

$$w^{\mathrm{T}} B = \alpha_1 (w_1^{\mathrm{T}} B) + \alpha_2 (w_2^{\mathrm{T}} B), \tag{75}$$

where it is noted that $w_1^{\mathrm{T}} B$ and $w_2^{\mathrm{T}} B$ are scalars as $m = 1$. Thus, there exists α_1 and α_2 such that $w^{\mathrm{T}} B = 0$. In this case,

$$w^{\mathrm{T}} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0, \tag{76}$$

implying that the matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \tag{77}$$

does not have full row rank (i.e., does not have rank n) and the system (70) is not controllable as a result of the Hautus test.

(10 points free)